SOME APPLICATIONS OF FIXED POINT THEOREMS

Synopsis of the thesis submitted in fulfillment of the requirements for the Degree of

DOCTOR OF PHILOSOPHY

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2012

1. INTRODUCTION

Let T be a self map of a nonempty set X. A point $p \in X$ such that $Tp = p$ is called a fixed point of the map T. However, if T is a multivalued map, i.e., from X to the collection of nonempty subsets of X, then a point p in X is called a fixed point of T if $p \in Tp$. The importance of the fixed point theory lies mainly in the fact that most of the equations arising in the various physical formulations may be transformed to fixed point equations or inclusions. The theorems concerning the properties and existence of fixed points are known as fixed point theorems. The roots of fixed point theory lie in the method of successive approximations for proving existence of solutions of differential equations introduced independently by Joseph Liouville [1] in 1837 and Charles Emile Picard [2] in 1890 (also see Granas and Dugundji [3], Kirk and Sims [4], Zeidler [5]). But formally its origin goes back to the beginning of twentieth century as an important part of nonlinear analysis. The abstraction of this classical theory is the pioneering work of the great Polish mathematician Stefan Banach [6] published in 1922 which provides a constructive method to find the fixed points of a map.

The Banach fixed point theorem states that *"a contraction mapping on a complete metric space has a unique fixed point"*. However, on historical point of view, the major classical result in fixed point theory is due to L. E. J. Brouwer [7] given in 1912, which states that "*a continuous map on a closed unit ball in R^n has a fixed point".* An extension of this result is the Schauder's fixed point theorem [8] of 1930 which states that "*a continuous map on a convex compact subspace of a Banach space has a fixed point".* Thereafter, many generalizations, extensions of these celebrated results enriched the theory of fixed points by a number of authors (see among others, Tychonoff [9], Kakutani [10], Lefschetz [11], Tarski [12], Edelstein [13], Kannan [14], Chatterjea [15], Zamfirescu [16], Ćirić [17] and references thereof). The Banach fixed point theorem forces the mapping T to be continuous on the space X. In 1968, Kannan [14] proved a fixed point theorem for operators that need not be continuous. Further, Chatterjea [15], in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. In this way, a lot of work has been reported in the literature on this line by using different classes of contraction type conditions. For an excellent comparisons of various contraction conditions, one may refer Rhoades [18-21], Collaco and Silva [22], Murthy [23] and Singh and Tomar [24]. The

extension of the theory of fixed points to multivalued maps was initiated by Kakutani [10] in the year 1941 for finite dimensional spaces. This was extended to infinite dimensional Banach spaces by Bohnenblust and Karlin [25] in 1950 and to locally convex space by Ky Fan [26] in 1952. Nadler, Jr. [27] in 1969 introduced the concept of multivalued contraction mappings motivated by the contraction principle of Banach.

Thus a huge development is reported in the study of fixed point theory of single valued, multivalued and hybrid maps in different directions. This in turn enhances the applications of fixed point theorems to diverse disciplines of mathematics, statistics, engineering and economics. In the present work, our attempt is to extend, improve and generalize several recent results and explore them for applicational purpose.

2. OBJECTIVES

The main objectives of the work reported in the thesis are:

- to obtain some approximate fixed point results and explore them for applications,
- to establish some common fixed point theorems and their applications,
- to study the role of fixed point theorems in the stability of iterative schemes,
- to use fixed point results for obtaining minimax and saddle point theorems,
- to study and extend the fixed point results for iterated function / multifunction systems.

3. WORK DONE

The thesis is organised chapter wise as follows:

Chapter 1

The first chapter is introductory in nature and provides the development of the subject and necessary background to the rest of the chapters in the thesis.

Chapter 2

In chapter 2, the concepts of approximate fixed points are discussed for various cases. In many situations of practical utility, the mapping under consideration may not have an exact fixed point due to some tight restriction on the space or the map, or an approximate fixed point is more than enough, an approximate solution plays an important role in such situations. This leads to the necessity of a proper theory regarding approximate fixed point. Tijs et al [28] obtained an approximate fixed point result in Banach spaces which is further extended for

several types of operators on metric spaces by Berinde [29]. The most general operator considered in that was the weak contraction which has been introduced in [30]. In this chapter, we obtain some basic approximate fixed point results in generalized metric spaces. Further, we establish some existence results concerning approximate fixed points, approximate common fixed points, approximate coincidence points, coincidence points and endpoints of multivalued contractions. We also develop quantitative estimates of the sets of approximate fixed points and approximate endpoints in generalized metric spaces.

We first obtain the following basic results in *b*-metric spaces:

Theorem 1. Let (X, d) be a *b*-metric space and $T : X \to X$ satisfy $d(Tx, Ty) \le \alpha\, d(x, y)$, $\alpha \in (0, 1)$. Then T has approximate fixed point property, i.e., for every $\varepsilon > 0$, $Fix_{\varepsilon}(T) \ne \phi$, where, $Fix_{\varepsilon}(T)$ is the set of all approximate fixed points of *T*.

Theorem 2. Let (X, d) be a *b*-metric space and $T : X \to X$ satisfy $d(Tx, Ty) \le \alpha\, d(x, y)$, $\alpha \in (0, 1)$. Then for each $\varepsilon > 0$, the diameter of $Fix_{\varepsilon}(T)$ is not larger than $b\varepsilon(1+b)/(1-\alpha b^2)$.

Similar more results are derived for the maps satisfying Kannan, Chatterjea, Zamfirescu, quasi contraction and almost or weak contraction conditions. These theorems extend the various results given by Tijs et al [28], Berinde [29], P˘acurar and P˘acurar [31] and Singh and Prasad [32].

Further, we obtain some approximate fixed point and coincidence point results for multivalued maps satisfying different contraction conditions. Following is one of the results:

Theorem 3. Let (X, d) be a *b*-metric space and $T : X \to Cl(X)$ satisfy multivalued contraction, i.e., there exists a number $\alpha \in (0, 1)$ such that $H(Tx, Ty) \le \alpha\, d(x, y)$, $\forall\, x, y \in X$. Then T has a fixed point provided, either (X, d) is compact and the function $f(x) = d(x, Tx)$ is lower semicontinuous or *T* is closed and compact.

The following theorem ensures the existence of approximate coincidence point of two maps and includes the result of Hussain et al [33].

Theorem 4. Let (X, d) be a b-metric space and $T: X \to Cl(X)$ be a multivalued I-contraction, i.e, there exists a number $\alpha \in (0, 1)$ such that $H(Tx, Ty) \leq \alpha\, d(Ix, Iy)$, $\forall\, x, y \in X$. Then T has the approximate coincidence point property, provided each Tx is I-invariant. Further, if (X, d) is compact and the function $f(x) = d(Ix, Tx)$ is lower semicontinuous, then I and T have a coincidence point.

Similar results are obtained for other multivalued contractions also.

Next we extend the results of Hussain et al [33] and Amini-Harandi [34], regarding the existence of unique endpoint.

Theorem 5. Let (X, d) be a complete b-metric space and $I: X \to X$ be a continuous single-valued mapping such that $r\, d(x, y) \leq d(Ix, Iy)$, where $r > 0$ is a constant. Let $T: X \to Cl(X)$ be lower semicontinuous and satisfy multivalued I-contraction. Then I and T have a unique endpoint if and only if I and T have the approximate end point property.

Similar results are obtained for other multivalued contractions as well.

Mohsenalhosseini et al [35] define the concept of approximate best proximity pair that is stronger than the best proximity pair and prove some existence theorems for approximate best proximity pair. Inspired from them, we obtain following existence results of approximate best proximity pair:

Theorem 6. Let A and B be nonempty subsets of a b-metric space X. Suppose that the mapping $T: A \bigcup B \to A \bigcup B$ satisfies $T(A) \subseteq B,\ T(B) \subseteq A$ and

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = d(A, B) \text{ for some } x \in A \bigcup B,$$

then the pair (A, B) is an approximate best proximity pair.

Theorem 7. Let A and B be nonempty subsets of a b-metric space X and suppose that the mapping $T: A \bigcup B \to A \bigcup B$ satisfies $T(A) \subseteq B,\ T(B) \subseteq A,$ and

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) + \phi(d(A, B)),$$

for all $x \in A$ and $y \in B$, where $\phi, \psi \in \Phi$, which is the class of the functions $\phi, \psi: [0, \infty) \to [0, \infty)$ satisfying:

(i) ϕ and ψ are continuous and monotone nondecreasing,

(ii) $\phi(t) = 0$ *iff* $t = 0$, $\psi(t) = 0$ *iff* $t = 0$,

then the pair (A, B) is an approximate best proximity pair.

Theorem 8. Let A and B be nonempty subsets of a b-metric space X and suppose that the mapping $T: A\bigcup B \to A\bigcup B$ satisfies $T(A) \subseteq B,\ T(B) \subseteq A$ and

$$[d(Tx, Ty)]^3 \le c_1\, d(x, y)[d(y, Ty)]^2 + c_2\, d(x, Tx)[d(Tx, Ty)]^2$$
$$+ c_3\, d(y, T^2x)[d(Tx, T^2x)]^2 + c_4\, d(A, B)[d(y, T^2x)]^2,$$

for all $x, y \in A\bigcup B$ and $c_i \in R$ such that $\sum_{i=1}^{4} c_i < 1.$ Then the pair (A, B) is an approximate best proximity pair.

As an application of Theorem 7 we obtain an existence result for Hammerstein integral equation as follows:

Theorem 9. Let $K(t, \tau)$ satisfy $\left(\int_a^b K^2(t, \tau)\, d\tau\right)^{1/2} \le M < \infty$, where M is a positive constant and $g(t) \in C[a, b].$ Also let $H(\tau, x)$ satisfy a uniform Lipschitz condition with respect to its second argument, i. e., $\|H(\tau, x) - H(\tau, y)\| \le h\|x - y\|$ for every $\tau \in [a, b],$ $h > 0$ and $x, y \in R.$ Further, assume that $M\, h(b-a)^{1/2} < 1.$ Then there exists a solution $f(t) \in C[a, b]$ and an $\varepsilon > 0,$ such that

$$\left| f(t) - g(t) - \int_a^b K(t, \tau) H(\tau, x(\tau))\, d\tau \right| < \varepsilon.$$

Chapter 3

The intent of chapter 3 is to obtain some coincidence point and common fixed point theorems in different spaces. The existence and approximation results for common fixed points of families of mappings have been studied by various authors (see, for example [36-39]), mostly by relaxing the contraction condition of the map or sometimes by relaxing the condition on the space or both.

First we obtain common fixed point theorems for hybrid pair of maps satisfying an integral inequality and common property $(E.\ A.)$ in $b-$metric spaces, which modifies the results of Liu et al [38].

Definition 1 [38]. Let $f, g: X \to X$ and $F, G: X \to CB(X).$ The maps pair (F, f) and (G, g) are said to satisfy the common property $(E.\ A.)$ if there exist two sequences $\{x_n\}, \{y_n\}$ in X, some t in X, and A, B in $CB(X)$ such that

$$\lim_{n\to\infty} Fx_n = A,\ \lim_{n\to\infty} Gy_n = B,\ \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A\bigcap B.$$

Theorem 10. Let (X, d) be a complete $b-$metric space and let $f, g : X \to X$ and $F, G : X \to CB(X)$ such that

(i) $FX \subset gX$, $GX \subset fX$,

(ii) the pairs (F, f) and (G, g) satisfy the common property $(E.A)$,

(iii) for all $x, y \in X$, $\int_0^{H(Fx, Gy)} \phi(t)\,dt \le q\left(\int_0^{M(x, y)} \phi(t)\,dt\right)$,

where $\phi : \Re^+ \to \Re^+$ is Lebesgue integrable function which is summable, non-negative and $\int_0^{\varepsilon} \phi(t)\,dt > 0$, for each $\varepsilon > 0$. Also let

$M(x, y) = \max\{d(fx, gy), d(Fx, fx), d(Gy, gy), [d(Fx, gy) + d(Gy, fx)]/2\}$ with

$qs < 1$, $\lambda s < 1$, where $\lambda = \max\{q, \frac{qs}{2-qs}\}$. If fX and gX are closed subspace of X, then

(1) f and F have a coincidence point.

(2) g and G have a coincidence point.

(3) f and F have a common fixed point provided that f is $F-$weakly commuting at u and $ffu = fu$ for $u \in C(f, F)$, where $C(f, F) = \{x : x$ is a coincidence point of f and $F\}$.

(4) g and G have a common fixed point provided that g is $G-$weakly commuting at v and $ggv = gv$ for $v \in C(g, G)$.

(5) f, g, F and G have a common fixed point provided (3) and (4) are true.

Again we obtain a common fixed point result for four self mappings in b-metric space.

Theorem 11. Let (X, d) be a complete b-metric space and $A, B, S, T : X \to X$ be such that

(i) $AX \subset TX$, $BX \subset SX$,

(ii) the pairs (A, S) and (B, T) are weakly compatible,

(iii) for all $x, y \in X$, $\int_0^{d(Ax, By)} \phi(t)\,dt \le q\left(\int_0^{M(x, y)} \phi(t)\,dt\right)$,

where $\phi : \Re^+ \to \Re^+$ is a Lebesgue integrable mapping which is summable, non-negative and satisfies $\int_0^{\varepsilon} \phi(t)\,dt > 0$, for each $\varepsilon > 0$ and

$$M(x, y) = \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), [d(Sx, By) + d(Ty, Ax)]/2\},$$

with $qs < 1, \lambda s < 1$, where $\lambda = \max\{q, \frac{qs}{2-qs}\}$. If TX or SX is closed, then A, B, S and T have a common fixed point.

As an application of Theorem 11, we obtain an existence and uniqueness result for the functional equations arising in dynamic programming. Let X and Y are Banach spaces and $S \subset X$ is the state space, $D \subset Y$, a decision space, $R = (-\infty, \infty)$ and $B(S)$ is the set of all bounded real valued functions. Then the functional equations arising in dynamic programming are given as follows:

$$f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \ x \in S, \tag{1}$$

$$g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), \ x \in S, \tag{2}$$

where $T : S \times D \to S$ and $H_i, F_i : S \times D \times R \to R, \ i = 1, 2.$

Theorem 12. Suppose that the following conditions are satisfied:

(i) H_i and F_i are bounded for i = 1, 2.

(ii) $$\int_0^{|H_1(x, y, h(t)) - H_2(x, y, k(t))|} \phi(t)\, dt \le q\left(\int_0^{M(h, k)} \phi(t)\, dt\right),$$

where,

$$M(h, k) = \max\{|T_2 h(t) - T_2 k(t)|, |T_1 h(t) - A_1 h(t)|, |T_2 k(t) - A_2 k(t)|, \frac{1}{2}[|T_1 h(t) - A_2 k(t)| + |T_2 k(t) - A_1 h(t)|]\}$$

for all $(x, y) \in S \times D, \ h, k \in B(S)$ and $t \in S$. A_i and T_i (i = 1, 2) are defined by

$$\begin{cases} A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \text{ for all } x \in S, \ h \in B(S), \ i = 1, 2. \\ T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \text{ for all } x \in S, \ k \in B(S), \ i = 1, 2. \end{cases}$$

(iii) For any sequence $\{k_n\} \subset B(S)$ and $k \subset B(S)$ with $\lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0,$

there exists $h_i \in B(S)$ such that $k = T_i h_i$ for i = 1 or i = 2.

(iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$, such that

$$A_1 h(x) = T_2 k_1(x), \ A_2 h(x) = T_1 k_2(x), \ x \in S.$$

(v) For any $h \in B(S)$ with $A_i h = T_i h \ (i = 1, 2)$, we have $T_i A_i h = A_i T_i h.$

Then the system of functional equations (1) and (2) has a unique common solution in $B(S)$.

Zadeh [40] in 1965 introduced the concept of fuzzy set to describe vagueness or uncertainty mathematically. Gogeun [41] generalized fuzzy set to $\mathcal{L}-$fuzzy set in 1967 and then Saadati et al [37] defined $\mathcal{L}-$fuzzy metric space. Recently, Khojasteh et al [42] reported a new notion of $\theta-$metric space defined as follows:

Definition 2 [42]. Let $\theta:[0,\infty)\times[0,\infty)\to[0,\infty)$ be a continuous map with respect to each variable. Then θ is called a $B-$action if and only if it satisfies the following conditions:

(i) $\theta(0,0)=0$ and $\theta(t,s)=\theta(s,t)$ for all $t,s\geq 0$,

(ii) $\theta(s,t)<\theta(u,v)$ if $s<u$ and $t\leq v$ or $s\leq u$ and $t<v$,

(iii) for each $r\in \mathrm{Im}(\theta)-\{0\}$ and for each $s\in(0,r]$, there exists $t\in(0,r]$ such that

$\theta(t,s)=r$, where $\mathrm{Im}(\theta)=\{\theta(s,t):s\geq 0,t\geq 0\}$,

(iv) $\theta(s,0)\leq s$, for all $s>0$.

Then $\theta-$metric space is defined as follows:

Definition 3 [42]. Let X be a nonempty set. A mapping $d_\theta:X\times X\to[0,\infty)$ is called a $\theta-$ metric on X with respect to $B-$ action, if d_θ satisfies:

(i) $d_\theta(x,y)=0$ *iff* $x=y$,

(ii) $d_\theta(x,y)=d_\theta(y,x)$, for all $x,y\in X$,

(iii) $d_\theta(x,y)\leq\theta(d_\theta(x,z),d_\theta(z,y))$, for all $x,y,z\in X$.

With $\theta-$ metric thus defined, (X,d_θ) is called a $\theta-$metric space.

Using the concept of $\theta-$metric space, we define $\theta-\mathcal{L}-$fuzzy metric space in the following way:

Definition 4. The 3-triplet $(X,\mathcal{M},\mathcal{T})$ is said to be a $\theta-\mathcal{L}-$fuzzy metric space, if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous $t-$norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2\times(0,\infty)$ with respect to $B-$action and $\theta\in\mathcal{M}$ satisfies the following conditions for every x,y,z in X and t,s in $(0,\infty)$:

(i) $\mathcal{M}(x,y,t)>_L 0_{\mathcal{L}}$,

(ii) $\mathcal{M}(x,y,t)=1_{\mathcal{L}}$ *for all* $t>0$, *iff* $x=y$,

(iii) $\mathcal{M}(x,y,t)=\mathcal{M}(y,x,t)$,

(iv) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, \theta(t, s))$,

(v) $\mathcal{M}(x, y, .) : (0, \infty) \to L$ is continuous and $\lim_{t\to\infty} \mathcal{M}(x, y, t) = 1_{\mathcal{L}}$.

Now we obtain the following common fixed point theorem in $\theta - \mathcal{L} -$ fuzzy metric spaces, which includes the results of Manro et al [39]:

Theorem 13. Let A, B, S, T, I and J be mappings from a complete $\theta - \mathcal{L} -$ fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself satisfies property (C), i.e., $\mathcal{M}(x, y, t) = C$, for all $t > 0$ implies $C = 1_{\mathcal{L}}$, and

(i) $AI(X) \subset T(X)$, $BJ(X) \subset S(X)$,

(ii) $$\int_0^{\mathcal{M}(AIx,\, BJy,\, \phi(t))} \varphi(s)\, ds \geq_L r\left(\int_0^{\mathcal{M}(x,\, y,\, t)} \varphi(s)\, ds \right),$$

where,

$\mathcal{M}(x, y, t) = \max\{M(Sx, Ty, t), M(AIx, Sx, t), M(BJy, Ty, t), [M(AIx, Ty, t) + M(BJy, Sx)]/2\}$,

$\phi(t)$ satisfies condition (ϕ) (i.e., $\phi(t) : [0, \infty) \to [0, \infty)$ is nondecreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, then $\phi(t) < t$ for all $t > 0$), $r : L \to L$ is a continuous function such that $r(t) >_L t$ for each $t = (t_1, t_2) \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, and for all $x, y \in X$. Suppose that one of A, B, S, T, I and J is complete and pairs (AI, S) and (BJ, T) are weakly compatible, then A, B, S, T, I and J have a unique common fixed point.

If the condition of completness of the space is relaxed, we obtain following result:

Theorem 14. Let A, B, S, T, I and J be mappings from a $\theta - \mathcal{L} -$ fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself satisfies (i), (ii) of Theorem 13 and property (C). Suppose that one of A, B, S, T, I and J is complete and pairs (AI, S) and (BJ, T) are weakly compatible, then A, B, S, T, I and J have a unique common fixed point.

Chapter 4

The stability of iterative procedures is discussed in chapter 4. In fixed point theory various iterative procedures are available for approximating fixed point of nonlinear equations. These iterative procedures are used according to their suitability. The theory regarding stability of

iterative procedures plays an important role in the study of numerical computations. Many physical problems can be expressed as fixed point equation $Tx = x$ and this equation is solved by some fixed point iteration procedure $x_{n+1} = f(T, x_n), n = 0, 1, 2, \ldots,$ which yields a sequence of points $\{x_n\}$ that converges to a fixed point x of T.

The stability of iterative procedure involving two mappings is studied by Singh et al [43] and Singh and Prasad [44]. We extend their results in the following theorems:

Theorem 15. Let (X, d) be a metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X and z be a coincidence point of T and S, i.e., $Sz = Tz = p.$ Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by Jungck-Ishikawa iteration converges to $p.$ Suppose $\{Sy_n\} \subset X$ and

$Ss_n = (1-\beta_n)y_n + \beta_n Ty_n,\ \ \varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ts_n),\ \ n \geq 0.$

If the pair (S, T) satisfies $d(Tx, Ty) \leq ad(Sx, Sy) + Ld(Sx, Tx),\ \ a \in (0, 1),\ L \geq 0,$ then

$$(i)\ \ d(p, Sy_{n+1}) \leq d(p, Sx_{n+1}) + \prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)\ d(Sx_0, Sy_0) + La\sum_{j=0}^{n}\alpha_j\beta_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\ d(Sx_i, Tx_i)$$

$$+ L\sum_{j=0}^{n}\alpha_j\prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\ d(Sz_i, Tz_i) + \sum_{j=0}^{n}\ \prod_{i=j+1}^{n}(1-\alpha_i + a\alpha_i)\varepsilon_j,$$

where the product is 1, when $j = n.$

Further,

(ii) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0.$

Theorem 16. Let (X, d) be a b-metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X and z be a coincidence point of T and S, i.e., $Sz = Tz = p.$ Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by Jungck-Mann iteration converges to $p.$ Suppose $\{Sy_n\} \subset X$ and define

$\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n),\ \ n \geq 0.$

If the pair (S, T) satisfies $d(Tx, Ty) \leq \varphi(d(Sx, Tx)) + a\,d(Sx, Sy),$ then

(i) $d(p, Sy_{n+1}) \leq bd(p, Sx_{n+1}) + b^{n+1}\prod_{i=0}^{n}(1-\alpha_i + a\alpha_i)\ d(Sx_0, Sy_0)$

$$+\varphi\sum_{j=0}^{n}\alpha_j b^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i+a\alpha_i)\ d(Sx_i, Tx_i)+\sum_{j=0}^{n}b^{j+2}\prod_{i=j+1}^{n}(1-\alpha_i+a\alpha_i)\varepsilon_j.$$

Further,

(ii) $\lim_{n\to\infty} Sy_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0.$

Above results are further extended for Jungck-Mann and Jungck-Ishikawa iteration schemes by taking two metrics in *b*-metric spaces, which modifies the results of Olatinwo [45].

Now we establish some weak stability results for Jungck-Mann iteration process on the lines of Timis and Berinde [46] and Timis [47] in the settings of metric spaces by taking a general contraction condition.

Theorem 17. Let (X, d) be a metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y)\subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X and z be a coincidence point of T and S, i.e., $Sz = Tz = p$. Let $x_0 \in Y$ and the sequence $\{Sx_n\}$ generated by Jungck-Mann iteration converges to p. Suppose $\{Sy_n\}\subset X$ and define $\varepsilon_n = d(Sy_{n+1}, (1-\alpha_n)Sy_n + \alpha_n Ty_n),\ n\geq 0.$ If the pair (S, T) satisfies $d(Tx, Ty)\leq ad(Sx, Sy)+Ld(Sx, Tx),\ a\in(0, 1), L\geq 0,$ then the Jungck-Mann iteration is weak w^2-stable.

We have also obtained weak w^2-stability results for the Jungck-Ishikawa iteration process for pair of maps under different contraction conditions generalizing the results of Timis [47].

Chapter 5

In this chapter, we present a generalized version of the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem by extending the concept of Chang and Zhang [48] and Ansari et al [49]. As an application of it, a generalized minimax inequality is obtained and an existence result for the saddle point problem under general settings is derived. Consequently, a saddle point theorem for two person zero sum parametric game is also proved. Several existing well known results are obtained as special cases.

The KKM mapping is defined as follows:

Definition 5 [50]. Let E be a Hausdorff topological vector space and X be a nonempty subset of E. A multivalued mapping $G: X\to 2^E$, that is mapping with the values

$G(x) \subset E$, for each x in X, is called a *KKM* mapping if $\text{co}\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$ for each finite subset $\{x_1, x_2, \dots, x_n\} \subset X$, where $\text{co}\{x_1, x_2, \dots, x_n\}$ denotes the convex hull of the set $\{x_1, x_2, \dots, x_n\}$.

Definition 6 [48]. Let X be a nonempty subset of a Hausdorff topological vector space E. A multivalued mapping $G : X \to 2^E$ is called a generalized *KKM* mapping, if for any finite set $\{x_1, \dots, x_n\} \subset X$, there exists a finite subset $\{y_1, \dots, y_n\} \subset E$ such that for any subset $\{y_{i_1}, \dots, y_{i_k}\} \subset \{y_1, \dots, y_n\}, 1 \le k \le n,$ we have

$$\text{co}\{y_{i_1}, \dots, y_{i_k}\} \subset \bigcup_{j=1}^{k} G(x_{i_j}).$$

This definition can be extended for two maps in the following manner:

Definition 7. Let X be a nonempty subset of a Hausdorff topological vector space E and $F, G : X \to 2^E$. Then G is said to be a generalized *F-KKM* mapping if for any finite set $\{x_1, \dots, x_n\} \subset X$ and each $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, n\}$, we have $\text{co}\{\bigcup_{j=1}^{k} F(x_{i_j})\} \subset \bigcup_{j=1}^{k} G(x_{i_j})$.

First we prove the following result which is used in proving the *KKM* mapping theorem:

Theorem 18. Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $G : X \to 2^E$ be a multivalued mapping such that for each $x \in X$, $G(x)$ is finitely closed. Then the family of sets $\{G(x) : x \in X\}$ has a finite intersection property if and only if the mapping G is a generalized *F-KKM* mapping for some mapping $F : X \to 2^E$.

Following is a generalized version of the *KKM* mapping theorem:

Theorem 19. Let Y be a nonempty convex subset of a Hausdorff topological vector space E, $\phi \ne X \subset Y$ and maps $F, G : X \to 2^E$ with G an intersectionally closed valued map of Y. Furthermore, assume that there exists a nonempty subset $X_0 \subset X$, contained in some precompact convex subset Y_0 of Y, such that $G(x_0)$ is compact subset of Y for at least one $x_0 \in X_0$ and let $\overline{G} : X \to 2^E$ be a generalized *F-KKM* mapping. Then $\bigcap_{x \in X} G(x) \ne \phi$.

A general minimax inequality result is also obtained.

Theorem 20. Let X be a nonempty closed convex subset of a Hausdorff topological vector space E and $F: X \to 2^X$. Let $\gamma \in (-\infty, \infty)$ be a given number and maps $\phi, \psi : X \times X \to (-\infty, \infty)$ satisfy the following conditions:

(i) For any fixed $y \in X$, $\phi(x, y)$ is a $\gamma-$generally lower semicontinuous function in x.

(ii) For any fixed $x \in X$, $\psi(x, y)$ is a F-$\gamma-$generalized quasiconcave function in y.

(iii) $\phi(x, y) \le \psi(x, y)$ for all $(x, y) \in X \times X$.

(iv) The set $\{x \in X : \phi(x, y_0) \le \gamma\}$ is precompact for at least one $y_0 \in X$.

Then there exists $\bar{x} \in X$ such that $\inf_{x \in X} \sup_{z \in F(x)} \phi(\bar{x}, z) \le \gamma$.

Next we obtain a saddle point theorem in a Hausdorff topological vector space E.

Theorem 21. Let X be a nonempty closed convex subset of a Hausdorff topological vector space E. Let $\gamma \in (-\infty, \infty)$ be a given number, $F: X \to 2^X$ a surjective mapping and let $\phi : X \times X \to (-\infty, \infty)$ satisfy the following conditions:

(i) $\phi(x, y)$ is a $\gamma-$generally lower semicontinuous function in x and F-$\gamma-$generalized quasiconcave function in y.

(ii) $\phi(x, y)$ is a $\gamma-$generally upper semicontinuous function in y and F-$\gamma-$generalized quasiconvex function in x.

(iii) There exists $x_1, y_1 \in X$ such that the sets

$$\{x \in X : \phi(x, y_1) \le \gamma\} \text{ and } \{y \in X : \phi(x_1, y) \ge \gamma\} \text{ are precompact.}$$

Then, there exists a saddle point of $\phi(x, y)$, i.e., there exists $(\bar{x}, \bar{y}) \in X \times X$ such that

$$\phi(\bar{x}, y) \le \phi(\bar{x}, \bar{y}) \le \phi(x, \bar{y}), \text{ for all } x, y \in X.$$

Moreover, we also have $\inf_{x \in X} \sup_{y \in X} \phi(x, y) = \sup_{y \in X} \inf_{x \in X} \phi(x, y) = \gamma$.

Finally the following saddle point theorem for parametric games is obtained:

Theorem 22. Let A and B be two Hausdorff topological vector spaces and let $X \subset A$ and $Y \subset B$ be two nonempty closed convex subsets. Let $\gamma \in (-\infty, \infty)$ be a given number, and F be any surjective mappings and let $G_\theta = f - \theta g : X \times Y \to R$ satisfy the following conditions:

(i) $f(x, y)$ is a $\gamma-$generally lower semicontinuous function in x and F-$\gamma-$generalized quasiconvex function in y.

(ii) $f(x, y)$ is a $\gamma-$generally upper semicontinuous function in y and F-$\gamma-$generalized quasiconcave function in x.

(iii) $g(x, y)$ is a $\gamma-$ generally lower semicontinuous function in y and F-$\gamma-$generalized quasiconvex function in x.

(iv) $g(x, y)$ is a $\gamma-$ generally upper semicontinuous function in x and F-$\gamma-$generalized quasiconcave function in y.

(v) There exists $x_1, y_1 \in X$ such that the sets $\{x \in X : G_\theta(x, y_1) \leq \gamma\}$ and $\{y \in X : G_\theta(x_1, y) \geq \gamma\}$ are precompact.

Then, there exists a saddle point of $G_\theta(x, y)$, i.e., there exists $(\bar{x}, \bar{y}) \in X \times X$ such that $G_\theta(\bar{x}, y) \leq G_\theta(\bar{x}, \bar{y}) \leq G_\theta(x, \bar{y})$, for all $x, y \in X$.

Moreover, we also have $\inf_{x \in X} \sup_{y \in X} G_\theta(x, y) = \sup_{y \in X} \inf_{x \in X} G_\theta(x, y) = \gamma$.

Chapter 6

The purpose of chapter 6 is to establish and study a generalized iterated function system using the notion of generalized contractions. John E. Hutchinson [51] in 1981 defined a method for constructing fractals, called iterated function system (IFS). An IFS consists of a finite set of contraction mappings on a complete metric space. Further, Barnsley and Demko [52] studied and popularized the theory of iterated function systems. Rus and Triff [53] and Mate [54] replaced contraction constant by a comparison function to obtain their results. Petrusal [55] has replaced the Banach contraction by Meir-Keeler type operators [56] and obtained more general results. Recently A. Mihail and R. Miculescu [57] introduced the notion of generalized iterated function system (GIFS), which is a family of functions $f_1, \ldots, f_n : X^m \to X$, in a complete metric space and showed GIFS to be a natural generalization of the notion of IFS. Llorens-Fuster et al [58] defined mixed iterated function system by taking more general conditions and obtained a mixed iterated function system theory for $\phi-$contraction and Meir-Keeler contraction maps. We improve and extend their results in this chapter.

We have used following generalized definitions of contraction mappings:

Definition 8. A function $f : X^m \to X$, for each $x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_m \in X$, such that $x_i \neq y_i$ for some $i \in \{1, 2, \dots, n\}$, is said to be a:

(i) generalized $\phi-$contraction, if ϕ is a comparison function and we have,

$$d(f(x_1, x_2, \dots, x_m), f(y_1, y_2, \dots, y_m)) \leq \phi(\max\{d(x_1, y_1), d(x_2, y_2), \dots, d(x_m, y_m)\})$$

(ii) generalized a-contraction, if $a \in [0, 1)$ and we have,

$$d(f(x_1, x_2, \dots, x_m), f(y_1, y_2, \dots, y_m)) \leq a \max\{d(x_1, y_1), d(x_2, y_2), \dots, d(x_m, y_m)\}$$

where, contractivity factor is defined as follows:

$$a = \sup_{\substack{x_1, \dots, x_m ; y_1, \dots, y_m \\ \max\{d(x_1, y_1), \dots, d(x_m, y_m)\} > 0}} \frac{d(f(x_1, \dots, x_m), f(y_1, \dots, y_m))}{\max\{d(x_1, y_1), \dots, d(x_m, y_m)\}}$$

(iii) generalized Meir-Keeler contraction if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\max\{d(x_1, y_1), \dots, d(x_m, y_m)\} < \varepsilon + \delta \Rightarrow d(f(x_1, \dots, x_m), f(y_1, \dots, y_m)) < \varepsilon$$

(iii) generalized contractive if we have

$$d(f(x_1, x_2, \dots, x_m), f(y_1, y_2, \dots, y_m)) < \max\{d(x_1, y_1), d(x_2, y_2), \dots, d(x_m, y_m)\}.$$

Now we define generalized mixed iterated function system or GMIFS from X^m to X rather than contractions from X to itself.

Definition 9. Let (X, d) be a complete b-metric space and $m \in N$. A generalized mixed iterated function system or GMIFS on X of order m is defined by $S = (X, (f_k)_{k=1,n})$, consists of a finite family of functions $f_k : X^m \to X$, where $k = 1, 2, \dots, n$ such that $f_1, f_2, \dots, f_n$ are generalized $\phi-$contraction or generalized Banach contraction or generalized Meir-Keeler contraction or generalized contractive.

First we establish the following theorem, which extends the results of Meir and Keeler [56], M. Pacurar [59], and Suzuki [60].

Theorem 23. Let (X, d) be a complete b-metric space (with constant $b \geq 1$) such that the b-metric is a continuous functional and $f : X^m \to X$ be a generalized ϕ-contraction with ϕ a b-comparison function or generalized Meir-Keeler contraction. Moreover, for any

$x_0, x_1, \ldots, x_{m-1} \in X$, the sequence $(x_n)_{n\geq 1}$ defined by $x_{n+m} = f(x_n, x_{n+1}, \ldots, x_{n+m-1}), \ \forall n \in N$ has the property that $\lim_{n\to\infty} x_n = \alpha$. Then there exists a unique $\alpha \in X$ such that $f(\alpha, \alpha, \ldots, \alpha) = \alpha$.

The following result extends the one of the results of Petrusel [61].

Theorem 24. Let (X, d) be a complete *b*-metric space such that the *b*-metric is a continuous functional and $f_i : X^m \to X$, for $i \in \{1, 2, \ldots, m\}$ are operators satisfying the generalized $\phi-$contraction with ϕ a *b*-comparison function or generalized Meir-Keeler type condition. Then the fractal operator $T_f : (H(X), h) \to (H(X), h)$ defined by the relation $T_f(Y) = \bigcup_{i=1}^{m} f_i(Y, Y, \ldots, Y)$ is a generalized $\phi-$contraction or generalized Meir-Keeler type operator. Further, $FixT_f = \{A^*\}$ and $(T_f^n(A))_{n\in N}$ converges to A^*, for each $A_1, A_2, \ldots, A_m \in H(X)$.

Finally, the following existence and uniqueness result for mixed generalized multi iterated function system is proved:

Theorem 25. Let (X, d) be a complete *b*-metric space such that the *b*-metric is a continuous functional and $F_i : X^m \to H(X)$, for $i \in \{1, 2, \ldots, m\}$ are operators satisfying the generalized $\phi-$contraction with ϕ a *b*-comparison function or generalized Meir-Keeler type condition. Then the multivalued fractal operator $T_F : (H(X), h) \to (H(X), h)$ defined by the relation $T_F(Y) = \bigcup_{i=1}^{m} F_i(Y, Y, \ldots, Y)$ is a generalized $\phi-$contraction or generalized Meir-Keeler type operator. Further, $FixT_F = \{A^*\}$ and $(T_F^n(A))_{n\in N}$ converges to A^*, for each $A_1, A_2, \ldots, A_m \in H(X)$.

From this theorem, results of Llorens-Fuster et al [58], Petrusel [61], Lazar [62] and Boriceanu et al [63] are deduced.

Brief Summary

- In chapter 2, we extend and improve the results of Tijs et al [28], Berinde [29], P˘acurar and P˘acurar [31], Singh and Prasad [32], Hussain et al [33] and Amini-Harandi [34] concerning approximate fixed point, coincidence point, end point and approximate best proximity points. As an application of approximate best proximity pair theorem, we obtain the solution of the Hammerstein integral equation.
- In chapter 3, we establish common fixed point theorems for the maps satisfying some integral type contraction conditions in *b*-metric space. Our result modifies the results of Liu et al [38]. An application of the result in solving the functional equations of dynamic programming is also provided. Further, we define generalized fuzzy metric space and obtain some existence results as applications to fuzzy set theory. Our results also generalize the results of Manro et al [39].
- Chapter 4 discusses the role of fixed point theorems in the stability of iterative schemes of maps satisfying some general condition. Our results extend and improve the results of Singh et al [43], Singh and Prasad [44], Olatinwo [45], Timis and Berinde [46] and Timis [47]. We analyze the convergence of various iterative schemes and obtain faster convergence towards the solution (fixed point) by replacing the iterative scheme. A comparative study of the different iterative schemes is also presented for some examples reported in the literature.
- The generalized version of Knaster-Kuratowski-Mazurkiewicz (KKM) maps is studied in chapter 5. Some variants of KKM theorems are derived which extend some existing results. As applications of the results to game theory, we establish minimax theorem and saddle point theorem for two-person-zero-sum game and two-person-zero-sum parametric game, which extends the results of Chang and Zhang [48] and Ansari et al [49].
- In chapter 6, we study the iterated function systems (IFS) and iterated multivalued function system (IMS) in general settings. We define generalized contraction conditions and the corresponding iterated function and multi function system. Some existence and uniqueness results are also obtained. This theory extends several recent results and enhances the scope of IFS and IMS. Our results of this chapter contain the recent results of Meir and Keeler [56], Llorens-Fuster et al [58], M. Pacurar [59], Suzuki [60], Petrusel [61], Lazar [62] and Boriceanu et al [63].

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